P.13: 7 (Ch.1 Sec.5 Ex.7)

$$|P(z)| = |a_0 + a_1 z + \cdots + a_n z^n|$$

$$= \left| z^n \left(\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \cdots + a_n \right) \right|$$

$$\leq |z|^n \left(\frac{|a_0|}{|z|^n} + \frac{|a_1|}{|z|^{n-1}} + \cdots + \frac{|a_{n-1}|}{|z|} + |a_n| \right)$$

a_i $(i = 1, 2, \cdots, n)$ are finite. For z sufficiently large

$(|z| > R)$, we can have

$$\frac{|a_0|}{|z|^n} + \frac{|a_1|}{|z|^{n-1}} + \cdots + \frac{|a_{n-1}|}{|z|} < |a_n|$$

This gives $\quad |P(z)| < 2\,|a_n|\,|z|^n$

P.16: 7 (Ch.1 Sec.6 Ex.7)

$$|\mathrm{Re}(2 + \bar{z} + z^3)| \leq |2 + \bar{z} + z^3|$$

$$\leq 2 + |\bar{z}| + |z^3|$$

$$\leq 2 + 1 + 1 = 4$$

P. 24 : 5 (Ch. 1 Sec. 9 Ex. 5)

(a) $LHS = e^{i\frac{\pi}{2}}\left(\sqrt{1^2+(-\sqrt{3})^2}\, e^{i\tan^{-1}\left(\frac{-\sqrt{3}}{1}\right)}\right)\left(\sqrt{(\sqrt{3})^2+1^2}\, e^{i\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)}\right)$

$= e^{i\frac{\pi}{2}}(2e^{-i\frac{\pi}{3}})(2e^{i\frac{\pi}{6}}) = 4e^{i(\frac{\pi}{2}-\frac{\pi}{3}+\frac{\pi}{6})}$

$= 4(\cos(\frac{\pi}{3}) + i\sin(\frac{\pi}{3})) = RHS$

(b) $LHS = 5e^{i\frac{\pi}{2}}\left(\sqrt{2^2+1^2}\, e^{i\tan^{-1}(\frac{1}{2})}\right)^{-1} = \frac{5}{\sqrt{5}}e^{i(\frac{\pi}{2}-\tan^{-1}(\frac{1}{2}))}$

$= \sqrt{5}\left(\cos\left(\frac{\pi}{2}-\tan^{-1}(\frac{1}{2})\right) + i\sin\left(\frac{\pi}{2}-\tan^{-1}(\frac{1}{2})\right)\right)$

$= \sqrt{5}\left(\sin(\tan^{-1}(\frac{1}{2})) + i\cos(\tan^{-1}(\frac{1}{2}))\right) = RHS$

(c) $LHS = \left((\sqrt{3}^2+1^2)\, e^{i\tan^{-1}(\frac{1}{\sqrt{3}})}\right)^6$

$= 2^6 (e^{i\frac{\pi}{6}})^6 = 2^6 e^{i\pi}$

$= 64(\cos(\pi) + i\sin(\pi)) = RHS$

(d) $LHS = \left(\sqrt{1^2+(-\sqrt{3})^2}\, e^{i\tan^{-1}(\frac{-\sqrt{3}}{1})}\right)^{-10}$

$= (2e^{-i\frac{\pi}{3}})^{-10} = 2^{-10} e^{-\frac{10}{3}i\pi}$

$= 2^{-10}(\cos(-\frac{10}{3}\pi) + i\sin(-\frac{10}{3}\pi))$

$= 2^{-10}(-\frac{1}{2} + i\frac{\sqrt{3}}{2}) = RHS$

P.24: 9 (Ch.1 Sec.9 Ex.9)

Let $S = 1 + z + z^2 + \cdots + z^n$

Then
$$S - zS = 1 + z + z^2 + \cdots + z^n - z - z^2 - \cdots - z^n - z^{n+1} = 1 - z^{n+1}$$

$$\Rightarrow S = \frac{1 - z^{n+1}}{1 - z}$$

Now let $z = e^{i\theta}$ and by de Moive's formula,

$$S = 1 + (\cos\theta + i\sin\theta) + (\cos 2\theta + i\sin 2\theta) + \cdots + (\cos(n\theta) + i\sin(n\theta))$$
$$= (1 + \cos\theta + \cdots + \cos(n\theta)) + i(\sin\theta + \cdots + \sin(n\theta))$$

We also have
$$S = \frac{1 - (e^{i\theta})^{n+1}}{1 - e^{i\theta}}$$
$$= \frac{1}{1 - e^{i\theta}} \cdot \frac{1 - e^{-i\theta}}{1 - e^{-i\theta}} - \frac{e^{i\theta(n+1)}}{1 - e^{i\theta}} \cdot \frac{1 - e^{-i\theta}}{1 - e^{-i\theta}}$$
$$= \frac{(1 - \cos\theta) + i\sin\theta}{(1 - \cos\theta)^2 + \sin^2\theta} - \frac{(1 - \cos\theta) + i\sin\theta}{(1 - \cos\theta)^2 + \sin^2\theta}\left(\cos((n+1)\theta) + i\sin((n+1)\theta)\right)$$

$$Re(S) = \frac{1 - \cos\theta}{2 - 2\cos\theta} - \frac{(1 - \cos\theta)\cos((n+1)\theta) - \sin\theta \sin((n+1)\theta)}{4\sin^2(\frac{\theta}{2})}$$
$$= \frac{1}{2} - \frac{\cos((n+1)\theta) - \cos(n\theta)}{4\sin^2(\frac{\theta}{2})}$$
$$= \frac{1}{2} + \frac{2\sin\left(\frac{(n+1)\theta + n\theta}{2}\right)\sin\left(\frac{(n+1)\theta - n\theta}{2}\right)}{4\sin^2(\frac{\theta}{2})}$$
$$= \frac{1}{2} + \frac{\sin((n+\frac{1}{2})\theta)}{2\sin\frac{\theta}{2}}$$

Note that $Re(S) = 1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta$

P.31: 6 (Ch.1 Sec. 11 Ex. 6)

Let $z^4 + 4 = 0$. Then $z^4 = -4 = 4^{i\pi}$

$$z_k = \sqrt{2}\, e^{i\left(\frac{\pi}{4} + \frac{2\pi k}{4}\right)}, \quad k = 0, 1, 2, 3 \text{ resp.}$$

Simplify and we obtain

$z_0 = 1 + i$ $\qquad$ $z_1 = -1 + i$

$z_2 = -1 - i$ $\qquad$ $z_3 = 1 - i$

So, $z^4 + 4 = \left((z - z_0)(z - z_1)\right)\left((z - z_2)(z - z_3)\right)$

$= (z^2 - 2z + 2)(z^2 + 2z + 2)$

P.43: 2 (Ch.2 Sec. 14 Ex. 2)

(a) $f(z) = (x + iy)^3 + (x + iy) + 1$

$= x^3 + 3x^2 iy + 3x(iy)^2 + (iy)^3 + x + iy + 1$

$= (x^3 - 3xy^2 + x + 1) + i(3x^2 y - y^3 + y)$

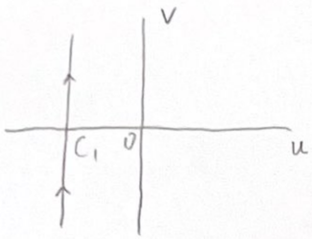
(b) $f(z) = \dfrac{(x - iy)^2}{x + iy} \cdot \dfrac{x - iy}{x - iy}$

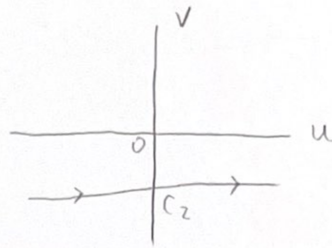
$= \dfrac{(x - iy)^3}{x^2 + y^2}$

$= \dfrac{x^3 + 3x(-iy)^2 + 3x^2(-iy) + (-iy)^3}{x^2 + y^2}$

$= \dfrac{x^3 - 3xy^2}{x^2 + y^2} + i\, \dfrac{-3x^2 y + y^3}{x^2 + y^2}$

P.43 : 6 (Ch.2 Sec. 14 Ex. 6)





(P. 40)

P.55 : 10 (Ch.2 Sec. 18 Ex. 10)

(a) $LHS = \lim_{\frac{1}{z} \to 0} \frac{4}{(1-\frac{1}{z})^2} = 4$

(b) $LHS = \lim_{z-1 \to 0} \frac{1}{(z-1)^3} = \infty$

(c) $\lim_{z \to 0} \frac{\frac{1}{z} - 1}{(\frac{1}{z})^2 + 1} = \lim_{z \to 0} \frac{1-z}{1+z^2} \cdot z = 0$

11.

$$\lim_{z \to 0} \frac{1}{T(\frac{1}{z})} = \lim_{z \to 0} \frac{c+dz}{a+bz} = \frac{c}{a}$$

(a) Obvious

(b) $\lim_{z \to -d/c} \frac{1}{T(z)} = \lim_{z \to -d/c} \frac{cz+d}{az+b} = 0$

So, $\lim_{z \to -d/c} T(z) = \infty \quad (c \neq 0)$

P. 108 : 9

(a) $|\sin z|^2 = \sin^2 x + \sinh^2 y \geq \sinh^2 y$

$$\downarrow$$

$$\leq 1 + \sinh^2 y = \cosh^2 y$$

Take square roots.

(b) $|\cos z|^2 = \cos^2 x + \sinh^2 y \geq \sinh^2 y$

$$\downarrow$$

$$\leq 1 + \sinh^2 y = \cosh^2 y$$

P. 111 : 2

(a) $\sinh 2z = \dfrac{e^{2z} - e^{-2z}}{2} = \dfrac{(e^z + e^{-z})(e^z - e^{-z})}{2} = 2 \sinh z \cdot \cosh z$

(b) $\sinh 2z = \sinh(i(-2iz)) = i \sin(-2iz)$

$$= 2i \sin(-iz)\cos(-iz)$$

$$= 2i(-i)\sinh(-i^2 z)\cosh(-i^2 z)$$

$$= 2 \sinh z \cosh z$$

P. 114 : 3

$$z = \cos^{-1}\sqrt{2} = \pm i \ln(\sqrt{2}+1) + 2n\pi, \quad n \in \mathbb{Z}$$